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A GENERALIZED MODE-MATCHING TECHNIQUE IN THE THEORY OF WAVEGUIDE MODE DIFFRACTION. PART 4: CONVERGENCE RATE OF THE PROJECTION APPROXIMATIONS *

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This part of the study continues presenting fundamentals of the generalized mode-matching technique designed for analyzing the waveguide mode scattering. Specifically, the problem is considered of analytically estimating the rate of convergence of the projection approximations to the operator Fresnel formulas the unconditional convergence of which has been proven earlier. By the way of example of a canonical scalar problem of wave diffraction by a step in a rectangular waveguide the derivation of the approximation error of the wave reflection and transmission operators is presented. It is shown that the formulated problem can be solved through considering the strong P-convergence of the projection representation of the amplitude scattering operator. As a result, an analytical estimate has been first obtained for the rate of convergence of the scattering operator approximations obtained using the truncation method for the operator Fresnel formulas. The found regularities are validated through numerical calculations. The obtained results make it possible to determine the computational efficiency of the generalized mode-matching technique.

KEY WORDS: *mode-matching technique, operator Fresnel formulas, rate of approximation convergence*

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1. INTRODUCTION

In the previous parts [1-3] of the study a theory of the generalized mode-matching technique has been developed on the basis of a new formulation of the diffraction problem for the waveguide modes. The new approach has allowed to i) rigorously prove the existence, uniqueness and stability of the solutions of the matrix-operator equations of the mode-matching technique for two classes of the electrodynamic analysis problems; ii) clarify that the correctness of the matrix model is a direct consequence of the energy conservation law; iii) prove the unconditional convergence of the projection approximations of the truncation technique to the true scattering operators, and iv) analytically estimate the condition number of the infinite and truncated matrices of the resultant model.

In this part of the study we consider a practically important problem of *a priori* estimating the quantitative parameters of convergence of the truncation technique approximations which was a formidable task within the standard mode-matching technique. Making use of the matrix operator technique, an analytical estimate is derived for the convergence rate of the approximations for the wave reflection $\mathbf{R}$ and transmission $\mathbf{T}$ operators using an example of the sane canonical scalar problem of diffraction of the $\{LM_{m0}\}_{m=1}^{\infty}$ and $\{LE_{m1}\}_{m=0}^{\infty}$ modes by a leap of the transverse cross-section of a rectangular waveguide which has been considered earlier in paper [1].

In the course of derivation of this estimate we will use the basic conceptions, terms and notation of the previous parts [1-3] of the study, and also the following definition of the order of approximation of the matrix operator (see, for example, reference [4]). For a given infinitesimally small numerical sequence $\left\{\alpha_N = N^{-\nu}\right\}_{N=1}^{\infty}$, with $\nu > 0$, the sequence of projection approximations $\left\{\widehat{\mathbf{R}}\right\}$ is P-convergent to the matrix operator $\mathbf{R}: \ell_2 \to \ell_2$ with the rate α_N, provided that

$$\left\| \mathbf{b}\left(\mathbf{P}\,\mathbf{R}\mathbf{P} - \widehat{\mathbf{R}}\right)\right\|_h \le \text{const} \cdot \alpha_N . \tag{1}$$

At that the order of approximation of the operator $\mathbf{R}$ on the vector $\mathbf{b}$ is equal to ν. In the inequality Eq. (1) $\mathbf{P}$ is a specified orthoprojector, while the norm is calculated in the space $h \equiv \mathbf{P}\ell_2$.

In the given part of study such approximations of the matrix operator appear which contain adjoint orthoprojectors $\mathbf{P}$ and $\mathbf{Q} = \mathbf{I} - \mathbf{P}$ $\left(\mathbf{PQ} = \mathbf{QP} = \mathbf{0}\right)$ alternating components of this operator in a certain order. This generalization of the standard reduction of the matrix operator will be regarded as a projection representation of the given operator even in the case where such an approximation is difficult to be attached with a direct physical meaning.

2. INEXACTNESS OF THE PROJECTION APPROXIMATIONS OF THE SCATTERING OPERATORS

The matrix model of the generalized mode-matching technique for the class of the problems of diffraction by a step discontinuity in a waveguide takes the form of the Fresnel formulas for the scattering operators [1]. In the case of a step in waveguide these formulas are as follows

$$\begin{cases} \mathbf{R}_p = \dfrac{\mathbf{D}_p - \mathbf{I}}{\mathbf{D}_p + \mathbf{I}}, \quad \mathbf{D}_p = \begin{Bmatrix} \mathbf{D}_0\mathbf{D}_0^T \\ \mathbf{D}_0^T\mathbf{D}_0 \end{Bmatrix}, \\ {}^{pq}\mathbf{T} = \left(\mathbf{D}_p + \mathbf{I}\right)^{-1} 2 \begin{Bmatrix} \mathbf{D}_0 \\ \mathbf{D}_0^T \end{Bmatrix}, \; p+q=3, \end{cases} \quad p = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}. \tag{2}$$

Here $\mathbf{R}_p$ represents (up to a sign) the operator of reflection in arm p, ${}^{pq}\mathbf{T}$ is essentially the operator of wave transmission from arm p into arm q, and the basic operator of the problem $\mathbf{D}_0$ is defined by the scalar product of transverse eigenfunctions of two partial domains and propagation constants of the waveguide modes [1].

It makes sense for our purposes to write the formulas Eq. (2) in terms of the pair of the following "amplitude operators of scattering"

$$\begin{cases} \mathbf{A}_p = \left(\mathbf{D}_p + \mathbf{I}\right)^{-1}, \\ \mathbf{B}_p = \dfrac{\mathbf{D}_p}{\mathbf{D}_p + \mathbf{I}}, \end{cases} \Leftrightarrow \begin{cases} \mathbf{B}_p + \mathbf{A}_p = \mathbf{I}, \\ \mathbf{B}_p - \mathbf{A}_p = \mathbf{R}_p, \end{cases} \tag{3}$$

to obtain

$$\begin{cases} \mathbf{R}_p = \mathbf{I} - 2\mathbf{A}_p = 2\mathbf{B}_p - \mathbf{I}, \\ {}^{pq}\mathbf{T} = 2\mathbf{A}_p \begin{Bmatrix} \mathbf{D}_0 \\ \mathbf{D}_0^T \end{Bmatrix}, \; p = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}. \end{cases} \tag{4}$$

The problem operator $\operatorname{Re}\mathbf{D}_p > 0$ is accretive. Whence it follows that the "amplitude operators of scattering" Eq. (3) are accretive contractions, $\operatorname{Re}\mathbf{A}_p > \mathbf{A}_p\mathbf{A}_p^\dagger > 0$ and $\operatorname{Re}\mathbf{B}_p > \mathbf{B}_p\mathbf{B}_p^\dagger > 0$ [1].

To construct projection approximations to the scattering operators Eq. (4) we will use the infinite-dimensional orthoprojectors [2]

$$\mathbf{P}_K \equiv \left\{ P_{mn}^{(K)} = \sum_{p=(0)1}^{K} \delta_{mp}\delta_{pn} \right\}, \quad \mathbf{Q}_K \equiv \mathbf{I} - \mathbf{P}_K, \tag{5}$$

where $K = M, N$ denotes the number of modes considered in two semi-infinite waveguides and δ_{mn} stands for the Kronecker delta. Next is supposed that the field in waveguide p, $p = 1, 2$, is reduced to a sum of M modes, while N modes are considered in the adjacent region.

The projection approximations under consideration,

$$\begin{cases} \widehat{\mathbf{R}}_p = \mathbf{P}_M - 2\widehat{\mathbf{A}}_p \mathbf{P}_M = \begin{bmatrix} \tilde{\mathbf{R}}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \\ {}^{pq}\widehat{\mathbf{T}} = 2\widehat{\mathbf{A}}_p \begin{Bmatrix} \widehat{\mathbf{D}}_0 \\ \widehat{\mathbf{D}}_0^T \end{Bmatrix} = \begin{bmatrix} {}^{pq}\tilde{\mathbf{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \ p = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} \end{cases} \tag{6}$$

represent extensions of the finite-dimensional approximations $\tilde{\mathbf{R}}_p$ and ${}^{pq}\tilde{\mathbf{T}}$ [2] to infinite-dimensional matrices using zeroes as shown symbolically in the expressions Eq. (6) with the use of the block matrices. The notation used here is as follows

$$\begin{gathered} \widehat{\mathbf{A}}_p = \left(\widehat{\mathbf{D}}_p + \mathbf{I}\right)^{-1}, \ \widehat{\mathbf{A}}_p \widehat{\mathbf{A}}_p^{-1} = \widehat{\mathbf{A}}_p^{-1} \widehat{\mathbf{A}}_p = \mathbf{I}, \\ \widehat{\mathbf{D}}_p = \begin{cases} \widehat{\mathbf{D}}_0 \widehat{\mathbf{D}}_0^T, \\ \widehat{\mathbf{D}}_0^T \widehat{\mathbf{D}}_0, \end{cases} \ \widehat{\mathbf{D}}_0 = \begin{Bmatrix} \mathbf{P}_M \\ \mathbf{P}_N \end{Bmatrix} \mathbf{D}_0 \begin{Bmatrix} \mathbf{P}_N \\ \mathbf{P}_M \end{Bmatrix}, \ p = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}. \end{gathered} \tag{7}$$

It has been found in paper [2] that the strong P-convergence of the projection approximations Eq. (6) to the true operators of scattering Eq. (4) is determined by the strong P-convergence of the difference of the known operators $\mathbf{P}_M \mathbf{D}_p - \widehat{\mathbf{D}}_p = \mathbf{\Lambda}_{M,N}^{(p)}$ to the null operator. For the problem under consideration this difference can be represented in the following form

$$\mathbf{\Lambda}_{M,N}^{(p)} = \mathbf{P}_M \mathbf{D}_p \mathbf{Q}_M + \mathbf{P}_M \begin{Bmatrix} \mathbf{D}_0 \\ \mathbf{D}_0^T \end{Bmatrix} \mathbf{Q}_N \begin{Bmatrix} \mathbf{D}_0^T \\ \mathbf{D}_0 \end{Bmatrix} \mathbf{P}_M. \tag{8}$$

To investigate the P-convergence of the scattering operators, we produce the differences

$$\mathbf{P}_M \mathbf{R}_p \mathbf{P}_M - \widehat{\mathbf{R}}_p = 2\widehat{\mathbf{A}}_p \mathbf{\Lambda}_{M,N}^{(p)} \mathbf{A}_p \mathbf{P}_M; \tag{9}$$

$$\mathbf{P}_M\,{}^{pq}\mathbf{T}\,\mathbf{P}_N - {}^{pq}\widehat{\mathbf{T}} = -\widehat{\mathbf{A}}_p \boldsymbol{\Lambda}_{M,N}^{(p)}\,{}^{pq}\mathbf{T}\,\mathbf{P}_N. \tag{10}$$

Then with account of the properties $\left\|\mathbf{A}_p\right\| < 1$ and $\left\|{}^{pq}\mathbf{T}\right\| < \sqrt{2}$ [5] the equalities Eq. (9) and (10) yield the estimate

$$\left.\begin{array}{l}\left\|\mathbf{b}\left(\mathbf{P}_M \mathbf{R}_p \mathbf{P}_M - \widehat{\mathbf{R}}_p\right)\right\| \\ \left\|\mathbf{b}\left(\mathbf{P}_M\,{}^{pq}\mathbf{T}\,\mathbf{P}_N - {}^{pq}\widehat{\mathbf{T}}\right)\right\|\end{array}\right\} < \left\|\mathbf{a}_1\mathbf{Q}_M\right\| + \left\|\mathbf{a}_2\mathbf{Q}_N\right\|, \tag{11}$$

where $\mathbf{a}_1, \mathbf{a}_2 \in \ell_2$ with $\forall \mathbf{b} \in \ell_2$ [2].

So, the strong P-convergence of the projection approximations Eq. (6) with any values of the ratio M/N is a corollary fact of the strong convergence of the orthoprojector $\mathbf{Q}_K$, with $K = M, N$, to the null operator in the space ℓ_2. The presence of two terms in the expression Eq. (11) implies the necessity of simultaneous and independent fulfillment of the limiting passage conditions $M \to \infty$ and $N \to \infty$.

It will be shown below that the convergence rate of the approximations Eq. (6) can be estimated using properties of the operator

$$\boldsymbol{\Upsilon}_{M,N}^{(p)} = 2\mathbf{B}_p\mathbf{Q}_M + {}^{pq}\mathbf{T}\mathbf{Q}_N \begin{Bmatrix}\mathbf{D}_0^T \\ \mathbf{D}_0\end{Bmatrix}\mathbf{P}_M, \tag{12}$$

which also can be represented in the following equivalent form

$$\boldsymbol{\Upsilon}_{M,N}^{(p)} = {}^{pq}\mathbf{T}\left[\begin{Bmatrix}\mathbf{D}_0^T \\ \mathbf{D}_0\end{Bmatrix} - \mathbf{P}_N \begin{Bmatrix}\mathbf{D}_0^T \\ \mathbf{D}_0\end{Bmatrix}\mathbf{P}_M\right]. \tag{13}$$

It is clear from the last expression that this operator is essentially a projection representation of the amplitude operator $2\mathbf{B}_p$, $\left\|\mathbf{B}_p\right\| < 1$.

It follows from the expansion of the operator $\boldsymbol{\Upsilon}_{M,N}^{(p)}$ in adjoint orthoprojectors Eq. (12) that the Pythagorean theorem holds for an arbitrary vector $\mathbf{b} \in \ell_2$, viz.

$$\left\|\mathbf{b}\boldsymbol{\Upsilon}_{M,N}^{(p)}\right\|^2 = \left\|2\mathbf{b}\mathbf{B}_p\mathbf{Q}_M\right\|^2 + \left\|\mathbf{b}\,{}^{pq}\mathbf{T}\mathbf{Q}_N \begin{Bmatrix}\mathbf{D}_0^T \\ \mathbf{D}_0\end{Bmatrix}\mathbf{P}_M\right\|^2. \tag{14}$$

Let the field source vector $\mathbf{b}$ is such that $\mathbf{d}_1 = \mathbf{b}\mathbf{B}_p\mathbf{Q}_M$ and $\mathbf{d}_2 = \mathbf{b}\,{}^{pq}\mathbf{T}\mathbf{Q}_N$, with $\mathbf{d}_1, \mathbf{d}_2 \in \ell_2$ being essentially the coefficient vectors of the field expansion within the aperture of the irregularity under consideration. The power law of decreasing these coefficients with great values of the indices is familiar [6]. It is determined by the geometry of the sharp edge of the step and is independent of the scattered wave. In the case under consideration of a perfectly conducting (metal) rectangular wedge we have

$$d_m^{(1)}, d_m^{(2)} = O\left(m^{-7/6}\right), m >> 1.$$

Now, making use of the asymptotic estimation for the residual series

$$\sum_{m=K+1}^{\infty}\left(\frac{\text{const}}{m^{7/6}}\right)^2 = \frac{\text{const}_0^2}{K^{4/3}}\left[1+O\left(K^{-1}\right)\right],\ K >> 1, \tag{15}$$

the equality Eq. (14) can be rewrite in the form

$$\left\|\mathbf{b}\Upsilon_{M,N}^{(p)}\right\|^2 = \frac{\text{const}_1^2}{M^{4/3}} + \frac{\text{const}_2^2}{N^{4/3}} + O\left(M^{-7/6}\right) + O\left(N^{-7/6}\right), \tag{16}$$

and hence, $\lim\limits_{M,N\to\infty}\left\|\mathbf{b}\Upsilon_{M,N}^{(p)}\right\| = 0,\ \forall\mathbf{b}\in\ell_2$.

Lemma. The following identity is valid

$$2\widehat{\mathbf{A}}_p\mathbf{\Lambda}_{M,N}^{(p)} = \mathbf{P}_M\Upsilon_{M,N}^{(p)}\left(\mathbf{I}-\frac{1}{2}\Upsilon_{M,N}^{(p)}\right)^{-1}. \tag{17}$$

Proof. Making use of the expression Eq. (8) we obtain

$$2\widehat{\mathbf{A}}_p\mathbf{\Lambda}_{M,N}^{(p)} = \left(\mathbf{P}_M + \widehat{\mathbf{R}}'_p\right)\mathbf{Q}_M + {}^{pq}\widehat{\mathbf{T}}'\mathbf{Q}_N\begin{Bmatrix}\mathbf{D}_0^T\\ \mathbf{D}_0\end{Bmatrix}\mathbf{P}_M,\ p=\begin{Bmatrix}1\\ 2\end{Bmatrix}. \tag{18}$$

Here we have introduced the new projection representations for the scattering operators after the formulas

$$\begin{aligned}&\mathbf{P}_M + \widehat{\mathbf{R}}'_p = 2\widehat{\mathbf{A}}_p\mathbf{P}_M\mathbf{D}_p,\\ &{}^{pq}\widehat{\mathbf{T}}' = 2\widehat{\mathbf{A}}_p\mathbf{P}_M\begin{Bmatrix}\mathbf{D}_0\\ \mathbf{D}_0^T\end{Bmatrix},\ \ p=\begin{Bmatrix}1\\ 2\end{Bmatrix}.\end{aligned} \tag{19}$$

Next, it should be noted that the approximations $\widehat{\mathbf{R}}'_p$ and ${}^{pq}\widehat{\mathbf{T}}'$ are strongly *P*-convergent to the appropriate true operators of scattering. Indeed, the differences

$$\begin{aligned}&\mathbf{P}_M\mathbf{R}_p-\widehat{\mathbf{R}}'_p=-2\widehat{\mathbf{A}}_p\mathbf{\Lambda}^{(p)}_{M,N}\mathbf{B}_p,\\&\mathbf{P}_M\,{}^{pq}\mathbf{T}-{}^{pq}\widehat{\mathbf{T}}'=-\widehat{\mathbf{A}}_p\mathbf{\Lambda}^{(p)}_{M,N}\,{}^{pq}\mathbf{T}\end{aligned}\tag{20}$$

are quite similar to the equalities Eqs. (9) and (10) and hence, the estimates like Eq. (11) are applicable to these. Isolating these new projection representations of the scattering operators from the relations Eq. (20) and eliminating these from the equality Eq. (18), we arrive at the identity

$$2\widehat{\mathbf{A}}_p\mathbf{\Lambda}^{(p)}_{M,N}=\left(\mathbf{P}_M+\widehat{\mathbf{A}}_p\mathbf{\Lambda}^{(p)}_{M,N}\right)\mathbf{\Upsilon}^{(p)}_{M,N},\tag{21}$$

which can also be written in the form of Eq. (17).

Theorem. The projection approximations Eq. (6) are strongly *P*- convergent to the true scattering operators Eq. (4) with the rate $M^{-2/3}, N^{-2/3}$ with $M,N>>1$ for all the field source vectors $\mathbf{b}\in\ell_2$.

Proof. Substitution of the identity Eq. (17) into the equality Eq. (9) yields the estimate

$$\left\|\mathbf{b}\left(\mathbf{P}_M\mathbf{R}_p\mathbf{P}_M-\widehat{\mathbf{R}}_p\right)\right\|<\left\|\mathbf{b}\,\mathbf{P}_M\mathbf{\Upsilon}^{(p)}_{M,N}\right\|\left\|\left(\mathbf{I}-\frac{1}{2}\mathbf{\Upsilon}^{(p)}_{M,N}\right)^{-1}\right\|.$$

The second factor in the right-hand part of this expression represents a bounded quantity and its dependence on the values M and N can be disregarded. Next, for any finite-dimensional vector $\mathbf{bP}_M$, with $\mathbf{b}\in\ell_2$ we can use the result Eq. (16). Then on the condition $M,N>>1$ we obtain the inequality

$$\left\|\mathbf{b}\left(\mathbf{P}_M\mathbf{R}_p\mathbf{P}_M-\widehat{\mathbf{R}}_p\right)\right\|\le\begin{cases}\dfrac{|\mathrm{const}_3|}{M^{2/3}}\sqrt{1+\mathrm{const}_4^2\left(\dfrac{M}{N}\right)^{4/3}},\\[2ex]\dfrac{|\mathrm{const}_5|}{N^{2/3}}\sqrt{1+\mathrm{const}_6^2\left(\dfrac{N}{M}\right)^{4/3}}.\end{cases}$$

So, the order of approximation of the reflection operator is equal to $\nu=2/3$. On the same assumption the expression Eq. (10) yields the same estimate for the convergence rate of the wave transmission operator ${}^{pq}\mathbf{T}$.

Presented in Fig. 1 below are the typical results of computer verification of the obtained analytical estimate of the approximation order of the reflection operator $\mathbf{R}_1$ for the problem on a step within the *H*-plane. The numerical data are presented such that the value $-\nu$ to correspond to the slope ratio k of the straight-line equation. The computational results correspond to the values $a/\lambda=1.3$, $b/a=0.5$, $M/N=\{2;1/2\}$ and a specified field source vector $\mathbf{b}=\{\delta_{1m}\}_{m=1}^{\infty}$.

In the course of the computations the true reflection operator $\mathbf{R}_1$ was assumed to be represented by a finite $M_0 \times M_0$ matrix derived from the reduced model Eq. (41) (see paper [1]) for the values $M_0 = 4\,000$ and $N_0 = 2\,000$ $(M_0 / N_0 = a/b)$. In this condition the value of the condition number of the reversible 6000 × 6000 block matrix did not exceed the value 1.75.

As can be seen from the Figure, with relatively small values of the quantity N the deviations of the computed values from the interpolating straight line are somewhat greater for line 2 ($M/N = 1/2$) than for the ratio $M/N = 2$ which meets the Mittra rule (line 1).

The slope ratio of curve 1 in the Figure is equal to $k = -0.638$, while that of curve 2 is $k = -0.675$.

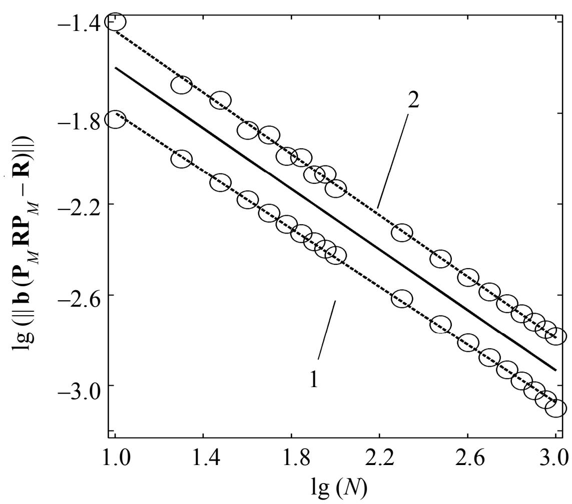

FIG. 1: Dependence of the reflection operator approximation error on the number of the considered waveguide modes. The circles correspond to the computer calculations, the dashed line represents the interpolating straight line and the solid line corresponds to the analytical estimation

Thus, the value of the relative error of the predicted value $\nu = 2/3$ for the graphs presented in the Figure does not exceed 4.5 %.

3. CONCLUSIONS

By way of example of the canonical problem on a step in a rectangular waveguide an analytical estimate has been first found for the rate of the strong P-convergence of the scattering operator approximations obtained through reduction of the operator Fresnel formulas.

As expected, the rate of convergence of the projection approximations under consideration is determined by the degree of decreasing the coefficients $\mathbf{d} = \{d_m\}$ of the mode expansion of the field within the waveguide irregularity aperture. It has been shown that the approximation order for the wave reflection **R** and transmission **T** operators within the problem under consideration is close to the value $\nu = 2/3$ with $d_m = O\left(m^{-7/6}\right)$, $m >> 1$, for all the field source vectors $\mathbf{b} \in \ell_2$.

The considered operator Fresnel formulas are of a universal character for the class of mode diffraction problems by step discontinuities in waveguides [1]. For this reason the suggested method of investigation and the obtained results would be useful for a rigorous analysis of discontinuities of the kind.

REFERENCES

1. Petrusenko, I.V. and Sirenko, Yu.K., (2013), Generalized mode-matching technique in the theory of guided wave diffraction. Part 1: Fresnel formulas for scattering operators, *Telecommunications and Radio Engineering*, **72**(5):369-384.
2. Petrusenko, I.V. and Sirenko, Yu.K., (2013), Generalized mode-matching technique in the theory of guided wave diffraction. Part 2: Convergence of projection approximations, *Telecommunications and Radio Engineering*, **72**(6):461-467.
3. Petrusenko, I.V. and Sirenko, Yu.K., (2013), Generalized mode-matching technique in the theory of guided wave diffraction. Part 3: Wave scattering by resonant discontinuities, *Telecommunications and Radio Engineering*, **72**(7):555-567.
4. Trenogin, V.A., (2002), *The functional analysis*, Fizmatlit, Moscow: 488 p. (in Russian).
5. Petrusenko, I.V. and Sirenko, Yu.K., (2009), Generalization of the power conservation law for scalar mode-diffraction problems, *Telecommunications and Radio Engineering*, **68**(16):1399-1410.
6. Mittra, R. and Lee, S.W., (1971), *Analytical techniques in the theory of guided waves*, The Macmillan Company, New York, - 327 p.